Numerical Estimation of the Curvature of Biological Surfaces

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Many biological systems may profitably be studied as surface phenomena. In this paper, we assume a model consisting of isotropic growth of a curved surface from a flat sheet. With such a model, the Gaussian curvature of the final surface determines whether growth rate of the surface is sub-harmonic or superharmonic. These properties correspond to notions of convexity and concavity, and thus to local excess growth and local deficiency of growth. In biological models where the major factors controlling surface growth are intrinsic to the surface, we have thus gained from geometrical study information on the differential growth undergone by the surface. We look at a specific application of these ideas to an analysis of the folding of the cerebral cortex, a geometrically rather complex surface growth.

The estimation of the curvature of biological surfaces requires a numerical technique which is robust in the presence of noise introduced by whatever digitisation procedure is used to quantify the surface. We develop a numerical surface curvature technique based on an approximation to the Dupin indicatrix of the surface. A metric for comparing curvature estimates is introduced, and considerable numerical testing indicates the reliability of this technique.

The curvatures of normal sections in different directions at a point on a surface are related by a quadratic form called the Dupin indicatrix
The eigenvalues of this form correspond to the maximum and minimum curvatures at that point on the surface, the eigenvectors correspond to the directions of maximum and minimum curvature. Gaussian and average curvatures are thus the product and mean respectively of these eigenvalues.

We estimate surface curvature in the following way. The curvature of a number of sections (not necessary normal) through a point is estimated from triplets of data points taken from a surface digitisation. The normal curvature in the same direction as each section is then estimated using Meusner's theorem. (If K is the normal curvature in the same direction as a section, angle ϕ from normal, and having curvature k, then $K = k \cos \phi$.) A least-squares approximation to the Dupin indicatrix is obtained from the normal curvatures and the principal values and directions of the indicatrix **are** used as estimates of surface curvature.

For the evaluation of an approximation procedure, it is important to have an appropriate measure of closeness for the quantity or quantities being approximated. A simple norm would suffice to measure Gaussian and average curvatures, which are scalar quantities. A full description of surface curvature, however, involves both the magnitude and direction of the principle curvatures. As these correspond to the eigenvalues and eigenvectors of the Dupin indicatrix, the following definition of a metric for examining surface curvature approximation error seems appropriate. Let L and M be two real symmetric 2 by 2 matrices representing the Dupin indicatrices of two surface curvatures. We define a metric d on the space of such matrices by $d(L,M) = \rho(L-M)$, where $\rho(L-M)$ is the spectral radius of L-M. This is then the spectral norm on the space of real symmetric matrices.

Having defined a suitable metric, we are able to evaluate the accuracy of our curvature estimation procedure in numerical tests using data from known surfaces with varying amounts of imposed noise. We compare the above method with one based on fitting an interpolant surface through points of the digitisation. In the presence of noise, the approach based on the Dupin indicatrix proves the more reliable.

An analysis of the surface curvature of the developing ferret brain reveals that regions of positive and negative Gaussian curvature coincide with the crests of developing folds (gyri) and their troughs (sulci) respectively. In the sense defined by the terms super- and sub-harmonic, gyri correspond to regions of differential excess growth, sulci to regions of differential growth deficiency. If the underlying growth processes are assumed to be intrinsic to the cortical surface, then we have achieved through analysis of the geometry of the brain a quantitative description of the differential growth of the cortex.